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Best Approximation over the Whole Complex Plane*

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Best rational approximation over the whole complex plane is investigated. While existence is elementary, there is not always uniqueness—every constant may be the best constant approximation to $f(z) = z$. However, under certain circumstances, the set of best approximations is, in a sense, bounded. When f has singularities of planar Lebesgue measure zero, the error corresponding to best approximation converges to zero, and the best approximations converge in measure.

1. MOTIVATION

If one forms a sequence of rational approximations with free (that is, unrestricted) poles, using only function values on a bounded set, it is well known that the sequence can diverge very badly outside this bounded set, even when the function approximated is entire. This is true whether the rational functions are formed by interpolation, or by best approximation in some norm. Hence it is of interest to study what happens when one uses function values throughout the plane. Here we form best approximations by minimizing an integral of a bounded distance function, over the whole plane. While best approximations exist, they are not unique in general, but the set of best approximations is "bounded" under certain circumstances. Further,

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provided the function approximated has singularities of measure zero, the error corresponding to best approximations converges to zero, and the best approximations converge in measure. A necessary condition for a polynomial to be a best approximation is established, and estimates of the error are obtained for certain functions.

2. Notation

(i) Throughout, meas will denote planar Lebesgue measure and μ a fixed (non-negative) regular Borel measure on the finite complex plane $\mathbb{C}$ such that

$$\int\int d\mu = \int\int_{\mathbb{C}} d\mu(z) = 1. \tag{2.1}$$

We shall assume that μ is absolutely continuous with respect to meas. The most interesting case is when $\operatorname{supp}[\mu] = \mathbb{C}$, but we do not exclude the case $\operatorname{supp}[\mu] \neq \mathbb{C}$.

(ii) Throughout $D(z, u)$ will denote a fixed function defined and continuous on $\mathbb{C} \times \mathbb{C}$ satisfying there:

$$\begin{aligned} &D(z, u) \in [0, 1], \\ &D(z, u) = D(u, z), \\ &D(z, u) = 0 \Leftrightarrow z = u. \end{aligned} \tag{2.2A}$$

We also assume that for each $z \in \mathbb{C}$,

$$D(z, \infty) = \lim_{\substack{|u| \to \infty \\ z_0 \to z}} D(z_0, u) \text{ exists and is positive.} \tag{2.2B}$$

Finally we set $D(\infty, \infty) = 0$. Corresponding to D, we define a distance between (Borel) measurable functions $f, g\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ by

$$\rho_D(f, g) = \int\int D(f, g)\, d\mu. \tag{2.3}$$

Of course, when D satisfies the triangle inequality in addition to (2.2A, B), then ρ_D also satisfies that inequality and is a metric on the space of all (equivalence classes of) functions $f\colon \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ that are finite valued a.e. (μ) and are (Borel) measurable.

(iii) An example of a $D(z,u)$ is $D_1(z,u)$, the square of the usual chordal metric on the Riemann sphere

$$D_1(z,u)=\frac{|z-u|^2}{(1+|z|^2)(1+|u|^2)}. \tag{2.4}$$

We put

$$\rho_1=\rho_{D_1}. \tag{2.5}$$

A second important example of $D(z,u)$ is

$$D_\Phi(z,u)=\Phi(|z-u|^2), \tag{2.6}$$

where

$$\begin{gathered}\Phi\colon [0,\infty]\to[0,1] \text{ is continuous and non-decreasing with}\\ 0=\Phi(0)<\Phi(x)<\Phi(\infty)=1 \qquad \text{for all } x\in(0,\infty).\end{gathered} \tag{2.7}$$

We set

$$\rho_\Phi=\rho_{D_\Phi}. \tag{2.8}$$

An important and typical case is $\Phi(u)=(u^\alpha/(1+u^\alpha))^\beta$, where $\alpha,\beta>0$. In the theory of Orlicz spaces, one encounters distances similar to ρ_Φ but with Φ convex and $\Phi(\infty)=\infty$; by contrast the Φ's that satisfy (2.7) are typically concave.

(iv) $\mathscr{R}_{mn}$ will denote the class of rational functions with complex coefficients and with numerator degree at most m and denominator degree at most n $(m,n=0,1,2,\ldots)$. Each $R\in\mathscr{R}_{mn}$, not identically zero, has the unique representation

$$R(z)=\frac{c\prod_{i=1}^{m'}(z-y_i)\prod_{i=m'+1}^{m''}(1-z/y_i)}{\prod_{i=1}^{n'}(z-z_i)\prod_{i=n'+1}^{n''}(1-z/z_i)}, \tag{2.9A}$$

where $c\neq 0$ and

$$\begin{array}{ll} 0\leqslant m'\leqslant m''\leqslant m; & 0\leqslant n'\leqslant n''\leqslant n;\\ |y_i|\leqslant 1 \quad \text{for all } 1\leqslant i\leqslant m'; & |y_i|>1 \quad \text{for all } m'<i\leqslant m'';\\ |z_i|\leqslant 1 \quad \text{for all } 1\leqslant i\leqslant n'; & |z_i|>1 \quad \text{for all } n'<i\leqslant n''; \end{array} \tag{2.9B}$$

and no y_i is a z_j.

The coefficient c in (2.9A) will be called the principal coefficient of R. The principal coefficient of 0 is 0. $\mathscr{P}_m$ will denote the class of polynomials of degree at most m with complex coefficients.

(v) The (Borel) measurable functions $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ considered will often be restricted to satisfy

$$\lim_{r\to\infty} \mu\{z: |f(z)| > r\} = 0 \qquad \text{or, equivalently,}$$
$$\mu\{z: |f(z)| = \infty\} = 0. \tag{2.10}$$

By saying a function f has singularities of meas 0 in an open set $\mathscr{C} \subset \mathbb{C}$ is meant that there is a closed set $\mathscr{E} \subset \mathbb{C}$ such that meas $(\mathscr{E}) = 0$ and such that f is analytic in each of the (at most denumerably many) components of $\mathscr{C} \backslash \mathscr{E}$.

(vi) Given a measurable $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ and $m, n = 0, 1, 2,\ldots$, define

$$E_{mnD}(f) = \inf\{\rho_D(f, R): R \in \mathscr{R}_{mn}\} \tag{2.11}$$

and

$$\mathscr{B}_{mnD}(f) = \{R \in \mathscr{R}_{mn}: \rho_D(f, R) = E_{mnD}(f)\}. \tag{2.12}$$

Thus $\mathscr{B}_{mnD}(f)$ is the set of ρ_D-best approximations to f from $\mathscr{R}_{mn}$. When $D = D_1$ or $D = D_\Phi$, we shall replace the subscript D in $E_{mnD}(f)$, $\mathscr{B}_{mnD}(f)$ by 1 or Φ, respectively.

When D (and hence ρ_D) satisfies the triangle inequality, we are dealing with best rational approximation in the linear metric space of all (equivalence classes of) measurable functions satisfying (2.10). See Albinus [1] for a survey of best approximation in real linear metric spaces.

3. Properties of ρ_D

In this section, we establish some properties of ρ_D. Many of these are trivial when D is a metric, but require proof in the general case where D does not satisfy the triangle inequality.

LEMMA 3.1. *Let $f, f_1, f_2,\ldots: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be measurable and let f satisfy* (2.10). *The following are equivalent.*

(i) *$f_k \to f$ in measure (μ): that is, for all $\varepsilon > 0$, $\lim_{k\to\infty} \mu\{z: |f(z) - f_k(z)| \geqslant \varepsilon\} = 0$.*

(ii) *Every subsequence of $\{f_k\}$ has a subsequence converging a.e. (μ) to f.*

(iii) $\lim_{k\to\infty} \rho_D(f_k, f) = 0$.

Proof. (i) $\Leftrightarrow$ (ii) is easy and well known.

(ii) $\Rightarrow$ (iii). Given any subsequence of $1, 2, \ldots$ we can extract from it a subsequence $\mathscr{S}$ so that $\lim_{k\to\infty, k\in\mathscr{S}} f_k(z) = f(z)$ a.e. (μ). Then $\lim_{k\to\infty, k\in\mathscr{S}} D(f_k(z), f(z)) = 0$ a.e. (μ) and so, by Lebesgue's Dominated Convergence Theorem, $\lim_{k\to\infty, k\in\mathscr{S}} \rho_D(f_k, f) = 0$. As every subsequence of $\{f_k\}$ contains such a subsequence $\{f_k\}_{k\in\mathscr{S}}$, the result follows.

(iii) $\Rightarrow$ (i). Let $\varepsilon, \eta > 0$ be given. Choose $r > 0$ such that

$$\mu\{z: |f(z)| \geqslant r/2\} < \eta. \tag{3.1}$$

As $D(z, u)$ is continuous in $\{(z, u): |z| \leqslant r, |u| \leqslant r\}$ and vanishes iff $z = u$, we see that there exists $\delta > 0$ such that

$$|z| \leqslant r, \qquad |u| \leqslant r \qquad \text{and} \qquad |z-u| \geqslant \varepsilon \Rightarrow D(z, u) \geqslant \delta. \tag{3.2}$$

Further (2.2B) gives

$$\alpha = \inf\{D(z, u): |z| \leqslant r/2 \text{ and } |u| \geqslant r\} > 0. \tag{3.3}$$

Finally,

$$\{z: |f(z) - f_k(z)| \geqslant \varepsilon\} \subset \mathscr{E}_k \cup \mathscr{U}_k \cup \mathscr{V}, \tag{3.4}$$

where

$$\mathscr{E}_k = \{z: |f(z)| \leqslant r, |f_k(z)| \leqslant r \text{ and } |f(z) - f_k(z)| \geqslant \varepsilon\},$$

$$\mathscr{U}_k = \{z: |f(z)| \leqslant r/2 \text{ and } |f_k(z)| \geqslant r\},$$

$$\mathscr{V} = \{z: |f(z)| \geqslant r/2\}.$$

Here

$$\begin{aligned} \mu(\mathscr{E}_k) &\leqslant \mu\{z: D(f(z), f_k(z)) \geqslant \delta\} \qquad \text{(by (3.2))} \\ &\leqslant \iint_{\{z: D(f(z), f_k(z)) \geqslant \delta\}} D(f(z), f_k(z))/\delta \, d\mu \\ &\leqslant (1/\delta)_{\rho_D}(f, f_k) \to 0 \qquad \text{as} \quad k \to \infty. \end{aligned}$$

Similarly,

$$\begin{aligned} \mu(\mathscr{U}_k) &\leqslant \mu\{z: D(f(z), f_k(z)) \geqslant \alpha\} \qquad \text{(by (3.3))} \\ &\leqslant (1/\delta)\rho_D(f, f_k) \to 0 \qquad \text{as} \quad k \to \infty. \end{aligned}$$

Finally $\mu(\mathscr{V}) < \eta$ by (3.1) so that (3.4) implies the result as $\eta > 0$ was arbitrary. Q.E.D.

We can now establish continuity of $\rho_D(f, g)$.

LEMMA 3.2. *Let $f, f_1, f_2, \ldots, g, g_1, g_2, \ldots: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be measurable and suppose that at least one of f, g satisfy* (2.10). *Then if* $\lim_{k\to\infty} \rho_D(f, f_k) = 0 = \lim_{k\to\infty} \rho_D(g, g_k)$, *we have* $\lim_{k\to\infty} \rho_D(f_k, g_k) = \rho_D(f, g)$.

Proof. Assume f satisfies (2.10). By Lemma 3.1, every subsequence of $\{f_k\}$ contains a subsequence $\{f_k\}_{k\in\mathscr{S}}$ such that a.e. (μ)

$$\lim_{\substack{k\to\infty \\ k\in\mathscr{S}}} f_k(z) = f(z). \tag{3.5}$$

Since $\lim_{k\to\infty} \rho_D(g, g_k) = 0$, we have $\lim_{k\to\infty} \mu\{z: D(g(z), g_k(z)) \geqslant \varepsilon\} = 0$, for all $\varepsilon > 0$, and hence $\mathscr{S}$ contains a subsequence (denoted also by $\mathscr{S}$ for convenience) such that a.e. (μ)

$$\lim_{\substack{k\to\infty \\ k\in\mathscr{S}}} D(g(z), g_k(z)) = 0. \tag{3.6}$$

Let $\mathscr{E} \coloneqq \{z: |g(z)| = \infty\}$. If $z \notin \mathscr{E}$, then, by (2.2A, B), $\lim_{k\to\infty, k\in\mathscr{S}} g_k(z) = g(z)$ provided (3.6) holds. Thus continuity of D and (3.5), (3.6) give

$$\lim_{\substack{k\to\infty \\ k\in\mathscr{S}}} D(f_k(z), g_k(z)) = D(f(z), g(z)) \qquad \text{for } \mu\text{-almost all } z \text{ in } \mathbb{C}\backslash\mathscr{E}. \tag{3.7}$$

If $z \in \mathscr{E}$ and (3.6) holds, we have, in view of (2.2B), that $\lim_{k\to\infty, k\in\mathscr{S}} |g_k(z)| = \infty$. Provided $|f(z)|$ is finite and (3.5) holds, we deduce $\lim_{k\to\infty, k\in\mathscr{S}} D(f_k(z), g_k(z)) = D(f(z), \infty) = D(f(z), g(z))$. Thus

$$\lim_{\substack{k\to\infty \\ k\in\mathscr{S}}} D(f_k(z), g_k(z)) = D(f(z), g(z)) \qquad \text{for } \mu\text{-almost all } z \text{ in } \mathscr{E}. \tag{3.8}$$

Using Lebesgue's Dominated Convergence Theorem, (3.7), (3.8) yield $\lim_{k\to\infty, k\in\mathscr{S}} \rho_D(f, g_k) = \rho_D(f, g)$. As every subsequence of 1, 2,... contains such a subsequence $\mathscr{S}$, the result follows. Q.E.D.

As a consequence of the above lemma, ρ_D-limits are unique: that is, if $\lim_{k\to\infty} \rho_D(f_k, f) = 0$ and $\lim_{k\to\infty} \rho_D(f_k, g) = 0$, and either f or g satisfies (2.10), then $f = g$ a.e. (μ).

LEMMA 3.3. *Let $f, f_1, f_2, \ldots: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be measurable and satisfy* (2.10). *The following are equivalent.*

(i) *There exists a measurable function f satisfying* (2.10) *such that* $\lim_{k\to\infty} \rho_D(f_k, f) = 0$.

(ii) $\{f_k\}$ *is fundamental in measure* (μ): *that is, for all* $\varepsilon > 0$, $\lim_{m\to\infty, k\to\infty} \mu\{z: |f_k(z) - f_m(z)| \geqslant \varepsilon\} = 0$.

(iii) $\lim_{m\to\infty, k\to\infty} \rho_D(f_k, f_m) = 0$ *and* $\lim_{r\to\infty} \sup_{k\geqslant 1} \mu\{z: |f_k(z)| \geqslant r\} = 0$.

Proof. (i) $\Rightarrow$ (ii). By Lemma 3.1, $f_k \to f$ in measure (μ). Hence [3, Theorem C, p. 92] $\{f_k\}$ is fundamental in measure (μ).

(ii) $\Rightarrow$ (i). By [3, Theorem E, p. 93] there is a measurable function f, finite valued a.e. (μ), such that $f_k \to f$ in measure (μ). Hence (2.10) holds. (Note that, although these two quoted theorems are stated for real functions, they are valid, without changes, for complex functions as well.)

(i) $\Rightarrow$ (iii). Let $1 \leqslant m(1) < m(2) < \cdots$ be integers, and set $g_k = f_{m(k)}$, $k = 1, 2, \ldots$. Then $\lim_{k\to\infty} \rho_D(g_k, f) = 0 = \lim_{k\to\infty} \rho_D(f_k, f)$, so Lemma 3.2 implies $\lim_{k\to\infty} \rho_D(g_k, f_k) = 0$. Since this held for any such $m(k)$, $k = 1, 2, \ldots$, we deduce $\lim_{m\to\infty, k\to\infty} \rho_D(f_k, f_m) = 0$. Next, given $\varepsilon > 0$, choose $r > 0$ such that $\mu\{z: |f(z)| \geqslant r\} \leqslant \varepsilon/2$ and choose $k_0 \geqslant 2$ such that $k \geqslant k_0 \Rightarrow \mu\{z: |f(z) - f_k(z)| \geqslant r\} \leqslant \varepsilon/2$. Then $k \geqslant k_0 \Rightarrow \mu\{z: |f_k(z)| \geqslant 2r\} \leqslant \mu\{z: |f_k(z) - f(z)| \geqslant r\} + \mu\{z: |f(z)| \geqslant r\} \leqslant \varepsilon$. By increasing r, if necessary, we can assume that the last inequality holds also for $k = 1, 2, \ldots, k_0 - 1$. The result follows.

(iii) $\Rightarrow$ (ii). Let $\eta, \varepsilon > 0$. Choose $r > 0$ such that

$$\sup_{k \geqslant 1} \mu\{z: |f_k(z)| \geqslant r\} < \varepsilon \tag{3.9}$$

and choose $\delta > 0$ such that

$$|z| \leqslant r, \qquad |u| \leqslant r \qquad \text{and} \qquad |z - u| \geqslant \eta \qquad \text{imply } D(z, u) \geqslant \delta. \tag{3.10}$$

Then

$$\begin{aligned} &\{z: |f_k(z) - f_m(z)| \geqslant \eta\} \\ &\quad \subset \{z: |f_k(z)| \geqslant r\} \cup \{z: |f_m(z)| \geqslant r\} \\ &\quad \cup \{z: |f_k(z)| \leqslant r, |f_m(z)| \leqslant r \text{ and } D(f_k(z), f_m(z)) \geqslant \delta\} \end{aligned}$$

by (3.10). So, by (3.9),

$$\begin{aligned} \mu\{z: |f_k(z) - f_m(z)| \geqslant \eta\} &< 2\varepsilon + \mu\{z: D(f_k(z), f_m(z)) \geqslant \delta\} \\ &\leqslant 2\varepsilon + (1/\delta)\rho_D(f_k, f_m) \to 2\varepsilon \qquad \text{as} \quad m, k \to \infty. \end{aligned}$$

Q.E.D.

In order to apply the preceding lemmas to rational functions, we need the following technical lemma.

LEMMA 3.4. *Let* $R \in \mathscr{R}_{mn}$ *have the representation* (2.9A, B). *Then, given* $A \geqslant 1 > \delta > 0$, *we have* $|c|\,(2A)^{-(m+n)}\,\delta^m \leqslant |R(z)| \leqslant |c|\,(2A)^{m+n}\,\delta^{-n}$ *whenever* $|z| \leqslant A$, $z \notin \mathscr{L}$, *where* $\text{meas}(\mathscr{L}) \leqslant 8e\pi\delta^2$. *If* $n = 0$, $\mathscr{L} = \phi$.

Proof. We can assume $c \neq 0$. Using (2.9A, B), we have, for $|z| \leqslant A$,

$$\left| c \prod_{i=1}^{m'} (z - y_i) \prod_{i=m'+1}^{m''} (1 - z/y_i) \right| \leqslant |c| \, (1+A)^m \leqslant |c| \, (2A)^m$$

and

$$\begin{aligned} |1 - z/z_i| &\geqslant 1/2 && \text{if} \quad |z_i| \geqslant 2A, \\ &\geqslant |z - z_i|/(2A) && \text{if} \quad |z_i| < 2A \end{aligned}$$

so, if $z_1 \cdots z_{n'}, z_{n'+1} \cdots z_l$ are those poles ζ of R with $|\zeta| < 2A$, then (2.9A) gives, whenever $|z| \leqslant A$,

$$\begin{aligned} |R(z)| &\leqslant \frac{|c| \, (2A)^m}{|\prod_{i=1}^{l} (z - z_i)| \, (1/(2A))^{l-n'} \, (1/2)^{n''-l}} \\ &\leqslant |c| \, (2A)^{m+n} \Big/ \left| \prod_{i=1}^{l} (z - z_i) \right|. \end{aligned}$$

By Cartan's lemma (see, for example, [2, pp. 174, 195]), $|\prod_{i=1}^{l} (z - z_i)| > \delta^l \geqslant \delta^n$ for all $z \notin \mathscr{L}_1$, where $\text{meas}(\mathscr{L}_1) \leqslant 4e\pi\delta^2$. Thus $|R(z)| \leqslant |c| \, (2A)^{m+n} \, \delta^{-n}$ whenever $|z| \leqslant A$, $z \notin \mathscr{L}_1$.

Replacing R by $1/R$, we have $|1/R(z)| \leqslant |1/c| \, (2A)^{m+n} \, \delta^{-m}$ whenever $|z| \leqslant A$, $z \notin \mathscr{L}_2$, where $\text{meas}(\mathscr{L}_2) \leqslant 4e\pi\delta^2$. By taking $\mathscr{L} = \mathscr{L}_1 \cup \mathscr{L}_2$, the result follows. Q.E.D.

LEMMA 3.5. *Let $\mathscr{F} \subset \mathscr{R}_{mn}$ be infinite. For each $R \in \mathscr{F}$, let $c(R)$ denote the principal coefficient of R. The following are equivalent.*

(i) $\sup\{|c(R)|: R \in \mathscr{F}\} < \infty$.

(ii) *Every infinite subset of $\mathscr{F}$ contains a sequence $\{R_k\}$ such that, for some $R \in \mathscr{R}_{mn}$, $\lim_{k \to \infty} \rho_D(R_k, R) = 0$.*

(iii) $\lim_{r \to \infty} \sup_{R \in \mathscr{F}} \mu\{z: |R(z)| \geqslant r\} = 0$.

(iv) *Whenever $\{R_k\} \subset \mathscr{F}$ satisfies $\lim_{k \to \infty} \rho_D(R_k, g) = 0$ for some measurable $g: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$, we have $g \in \mathscr{R}_{mn}$.*

Proof. Note, first that $\mathscr{R}_{mn}$ (and hence $\mathscr{F}$) is a normal family, that is, every infinite subset of $\mathscr{F}$ contains a sequence $\{R_k\}$ such that either for some $R \in \mathscr{R}_{mn}$, $\lim_{k \to \infty} R_k(z) = R(z)$ uniformly in each compact subset of $\mathbb{C} \backslash \{z: |R(z)| = \infty\}$ or $\lim_{k \to \infty} |R_k(z)| = \infty$ uniformly in each compact subset of $\mathbb{C} \backslash \mathscr{U}$, where $\mathscr{U}$ has at most m elements. This follows easily from (2.9A, B); the reader may also refer to [7, Theorem XII.1, p. 348]. Hence, every infinite subfamily of $\mathscr{F}$ contains a sequence $\{R_k\}$ such that $\lim_{k \to \infty} \rho_D(R_k, R) = 0$, where either $R \in \mathscr{R}_{mn}$ or $R \equiv \infty$ in $\mathbb{C}$.

(i) $\Rightarrow$ (ii). Given an infinite subset of $\mathscr{F}$, extract from it a sequence $\{R_k\}$ with limit R as above. Let $A \geqslant 1 > \delta > 0$, where $8e\pi\delta^2 < 1$. By Lemma 3.4, for all $k \geqslant 1$, $|R_k(z)| \leqslant \sup\{|c(R)|: R \in \mathscr{F}\}\,(2A)^{m+n}\,\delta^{-n}$ whenever $|z| \leqslant A$, $z \notin \mathscr{L}_k$, where $\operatorname{meas}(\mathscr{L}_k) \leqslant 8e\pi\delta^2 < 1 < \pi A^2$. Hence we cannot have $\lim_{k\to\infty} |R_k(z)| = \infty$, uniformly in every compact subset of $\{z: |z| \leqslant A\}\backslash\mathscr{S}$, where $\mathscr{S}$ is a finite set, so $R \not\equiv \infty$ and, hence, $R \in \mathscr{R}_{mn}$.

(ii) $\Rightarrow$ (iii). If (iii) fails, we can extract from $\mathscr{F}$ a sequence $\{R_k\}$ such that $\liminf_{r\to\infty} \sup_{k\geqslant 1} \mu\{z: |R_k(z)| \geqslant r\} > 0$. By passing again to a subsequence if necessary, we can (in view of (ii)) assume $\lim_{k\to\infty} \rho_D(R_k, R) = 0$ for some $R \in \mathscr{R}_{mn}$. Thus Lemma 3.3(i) can be applied and a contradiction to Lemma 3.3(iii) ensues.

(iii) $\Rightarrow$ (iv). Suppose $\lim_{k\to\infty} \rho_D(R_k, g) = 0$. By passing to a subsequence if necessary, we can assume $\lim_{k\to\infty} \rho_D(R_k, R) = 0$, where either $R \in \mathscr{R}_{mn}$ or $R \equiv \infty$. But for a suitable r, (iii) gives $\mu\{z: |R_k(z)| \geqslant r$ or $|z| \geqslant r\} \leqslant 1/2$ for all $k \geqslant 1$ so that $\mu\{z: |R_k(z)| \leqslant r$ and $|z| \leqslant r\} \geqslant 1/2$ for all such k. Thus we cannot have $\lim_{k\to\infty} |R_k(z)| = \infty$, uniformly in every compact subset of $\{z: |z| \leqslant r\}\backslash\mathscr{S}$, where $\mathscr{S}$ is a finite set. Hence $R \not\equiv \infty$ and $R \in \mathscr{R}_{mn}$. By Lemma 3.2, $g = R$ a.e. (μ), so $g \equiv R \in \mathscr{R}_{mn}$.

(iv) $\Rightarrow$ (i). If (i) were false, then we could choose $\{R_j\} \subset \mathscr{F}$ such that $|c(R_j)| \geqslant j^j$ for $j = 1, 2, \ldots$. By passing to a subsequence if necessary, we can assume $\lim_{j\to\infty} \rho_D(R_j, R) = 0$, where $R \in \mathscr{R}_{mn}$ or $R \equiv \infty$. Taking $A = j > 1$, $\delta = j^{-2}$, Lemma 3.4 gives $|R_j(z)| \geqslant j^j (2j)^{-(m+n)} j^{-2m}$ whenever $|z| \leqslant j$, $z \notin \mathscr{L}_j$ where $\operatorname{meas}(\mathscr{L}_j) \leqslant 8e\pi j^{-2}$. It follows that $\lim_{j\to\infty} |R_j(z)| = \infty$ for almost z (meas) in $\mathbb{C}$. Hence $R \equiv \infty$, which contradicts (iv). Q.E.D.

4. Existence of Best Approximations

THEOREM 4.1. *Let $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be measurable and satisfy* (2.10). *Let m, n, be nonnegative integers. If $E_{mnD}(f) < \rho_D(f, \infty)$, then*

(i) $\mathscr{B}_{mnD}(f) \neq \phi$.

(ii) $\sup\{|c(R)|: R \in \mathscr{B}_{mnD}(f)\} < \infty$, *where $c(R)$ denotes, as before, the principal coefficient of R.*

(iii) *$\mathscr{B}_{mnD}(f)$ is closed with respect to ρ_D, that is, if $\{R_k\} \subset \mathscr{B}_{mnD}(f)$ and $\lim_{k\to\infty} \rho_D(R_k, g) = 0$ for some measurable $g: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$, then $g \in \mathscr{B}_{mnD}(f)$.*

(iv) *$\mathscr{B}_{mnD}(f)$ is sequentially compact with respect to ρ_D, that is, every infinite subset of $\mathscr{B}_{mnD}(f)$ contains a sequence $\{R_k\}$ such that $\lim_{k\to\infty} \rho_D(R_k, R) = 0$ for some $R \in \mathscr{B}_{mnD}(f)$.*

(v) $\mathscr{B}_{mnD}(f)$ *is complete with respect to* ρ_D, *that is, if* $\{R_k\} \subset \mathscr{B}_{mnD}(f)$ *and* $\lim_{k\to\infty, m\to\infty} \rho_D(R_k, R_m) = 0$, *then* $\lim_{k\to\infty} \rho_D(R_k, R) = 0$ *for some* $R \in \mathscr{B}_{mnD}(f)$.

Proof. (i) Choose $\{R_k\} \subset \mathscr{R}_{mn}$ such that

$$\lim_{k\to\infty} \rho_D(R_k, f) = E_{mnD}(f). \tag{4.1}$$

By passing to a subsequence, if necessary, we can assume $\lim_{k\to\infty} \rho_D(R_k, R) = 0$, where $R \in \mathscr{R}_{mn}$ or $R \equiv \infty$ (as in the proof of Lemma 3.5). By Lemma 3.2 (with all $f_k = R_k$, all $g_k = f$), $\lim_{k\to\infty} \rho_D(R_k, f) = \rho_D(R, f)$ even when $R \equiv \infty$. Then $E_{mnD}(f) = \rho_D(R, f)$ (by (4.1)), contradicting $E_{mnD}(f) < \rho_D(f, \infty)$ if $R \equiv \infty$. Hence $R \in \mathscr{R}_{mn}$ and $R \in \mathscr{B}_{mnD}(f)$.

(ii) If this were false, then proceeding as in the proof of Lemma 3.5, (iv) $\Rightarrow$ (i), we could choose $\{R_i\} \subset \mathscr{B}_{mnD}(f)$ such that $\lim_{i\to\infty} |R_i(z)| = \infty$ for almost all z (meas) in $\mathbb{C}$. By passing to a subsequence, if necessary, we obtain $\lim_{i\to\infty} |R_i(z)| = |R(z)| \equiv \infty$, uniformly in every compact subset of $\mathbb{C} \backslash \mathscr{S}$ (where $\mathscr{S}$ is finite) and Lemma 3.2 gives $E_{mnD}(f) = \lim_{i\to\infty} \rho_D(f, R_i) = \rho_D(f, \infty)$, a contradiction.

(iii) If $\lim_{k\to\infty} \rho_D(R_k, g) = 0$ for $\{R_k\} \subset \mathscr{B}_{mnD}(f)$, then by Lemma 3.5(iv), $g \in \mathscr{R}_{mn}$ and Lemma 3.2 gives $\rho_D(g, f) = E_{mnD}(f)$; so $g \in \mathscr{B}_{mnD}(f)$.

(iv) Follows from Lemma 3.5(ii) much as (iii) followed from Lemma 3.5(iv).

(v) If $\{R_k\} \subset \mathscr{B}_{mnD}(f)$ and $\lim_{k\to\infty, m\to\infty} \rho_D(R_k, R_m) = 0$, then by (ii) above and Lemma 3.5(iii), $\lim_{r\to\infty} \sup_{k\geqslant 1} \mu\{z : |R_k(z)| \geqslant r\} = 0$. By Lemmas 3.3(iii) and (i), there exists a measurable g satisfying (2.10) such that $\lim_{k\to\infty} \rho_D(R_k, g) = 0$. By Lemma 3.5(iv), $g \in \mathscr{R}_{mn}$ and Lemma 3.2 shows $g \in \mathscr{B}_{mnD}(f)$. Q.E.D.

Remarks. (i) If one is prepared to regard $R(z) \equiv \infty$ as belonging to $\mathscr{R}_{mn}$, then the above shows that invariably $\mathscr{B}_{mnD}(f) \neq \phi$.

(ii) Some of the properties of $\mathscr{B}_{mnD}(f)$ stated in Theorem 4.1 are referred to as "approximative compactness" in the literature.

COROLLARY 4.2. *Let* D *satisfy* (2.6), (2.7), (2.8) *so that* $D = D_\Phi$. *Let* $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ *be measurable and satisfy* (2.10). *Then for all non-negative integers* m, n, $E_{mn\Phi}(f) < 1$, *and the conclusions of Theorem* 4.1 *hold.*

Proof. We have $D_\Phi(f(z), 0) = \Phi(|f(z)|^2) < 1 = \Phi(\infty) = D_\Phi(f(z), \infty)$ a.e. (μ) so $E_{mn\Phi}(f) < 1 = \rho_\Phi(f, \infty)$ for all $m, n \geqslant 0$. Q.E.D.

5. NON-UNIQUENESS

We show now that, given an integer $n \neq 0$, for $D = D_1$ every constant is a best constant approximation to $f(z) = z^n$ for a certain measure μ. This is rather disappointing, as one would expect that D_1—the square of the usual chordal metric on the Riemann sphere—would be the natural metric to use in best approximation over the whole complex plane.

LEMMA 5.1. *Let $f: \mathbb{C} \to \mathbb{C}$ be measurable and write $f(z) = |f(z)|\, e^{ih(z)}$ for all $z \in \mathbb{C}$. Then if*

$$I_1 = \iint (|f|^2 - 1)(|f|^2 + 1)^{-1}\, d\mu,$$

$$I_2 = -2 \iint |f|\, (\cos h)(|f|^2 + 1)^{-1}\, d\mu,$$

$$I_3 = -2 \iint |f|\, (\sin h)(|f|^2 + 1)^{-1}\, d\mu,$$

$$I_4 = \iint (|f|^2 + 1)^{-1}\, d\mu,$$

we have for each $s \geqslant 0$, $\theta \in [0, 2\pi)$, the equality $\rho_1(f, se^{i\theta}) = \{I_1 + s(I_2 \cos\theta + I_3 \sin\theta)\}/(1 + s^2) + I_4$. In particular, if $I_1 = I_2 = I_3 = 0$, then $\rho_1(f, u) = I_4$ for all $u \in \mathbb{C}$.

Proof.

$$\begin{aligned} |f(z) - se^{i\theta}|^2 &= \big||f(z)|\, e^{i(h(z)-\theta)} - s\big|^2 \\ &= (|f(z)|^2 - 1) - 2s\, |f(z)|\, \{(\cos h(z))(\cos\theta) \\ &\qquad + (\sin h(z))(\sin\theta)\} + (s^2 + 1). \end{aligned}$$

Dividing by $(|f(z)|^2 + 1)(s^2 + 1)$, we obtain $D_1(f(z), se^{i\theta})$ and integrating with respect to $d\mu$ gives the result. Q.E.D.

THEOREM 5.2. *Let $n \neq 0$ be an integer. Let*

$$\mu(\mathscr{S}) = (2\pi)^{-1} \iint_{\mathscr{S}} w(|z|)\, |z|^{-1}\, dx\, dy \qquad (x = \operatorname{Re} z, y = \operatorname{Im} z)$$

for all Borel sets $\mathscr{S}$, where w is an integrable non-negative function on $(0, \infty)$ satisfying

$$\int_0^\infty w(r)\, dr = 1 \qquad \text{and} \qquad \int_0^\infty \frac{r^{2n} - 1}{r^{2n} + 1}\, w(r)\, dr = 0 \tag{5.1}$$

($w(r)=|n|\,r^{2n-1}(r^{2n}+1)\exp(-r^{2n})$ satisfies (5.1)). Then for $f(z)=z^n$, we have $\rho_1(f,u)=E_{001}(f)$ for all $u\in\mathbb{C}$.

Proof. Using the notation of Lemma 5.1, we have $\cos(h(re^{i\theta}))=\cos n\theta$ for all real θ and all $r\geqslant 0$. Also

$$I_2=-2\int\int |f|\,(\cos h)(|f|^2+1)^{-1}\,d\mu$$

$$=-(1/\pi)\int\int_{\mathbb{C}} |f(z)|\,(\cos h(z))(|f(z)|^2+1)^{-1}\,w(|z|)\,|z|^{-1}\,dx\,dy$$

$$=-(1/\pi)\int_0^\infty\int_0^{2\pi} r^n(\cos n\theta)(r^{2n}+1)^{-1}\,w(r)\,d\theta\,dr$$

$(x=r\cos\theta;\,y=r\sin\theta)$

$$=-(1/\pi)\int_0^\infty \frac{r^n w(r)}{r^{2n}+1}\,dr\int_0^{2\pi}\cos n\theta\,d\theta=0.$$

Similarly $I_3=0$, while (5.1) gives $I_1=0$. Q.E.D.

Despite Theorem 5.2, we shall see that the elements of $\mathscr{B}_{mnD}(f)$ converge in measure (μ) to f as $\max\{m,n\}\to\infty$, so that $\mathscr{B}_{mnD}(f)$ eventually becomes "small"—in fact Theorem 4.1 shows that $\mathscr{B}_{mnD}(f)$ becomes "compact" as soon as $E_{mnD}(f)<\rho_D(f,\infty)$.

6. On Characterization of Best Polynomial Approximations

In attempting to characterize $\mathscr{B}_{m0D}(f)$, it seems natural to look for analogues of the well-known characterization of best polynomial approximations in L_p. For simplicity, we restrict ourselves to $D=D_\Phi$ in this section. Even for such D, we obtain only necessary conditions for a polynomial to belong to $\mathscr{B}_{m0\Phi}(f)$, because of the fact that Φ is not convex in general.

Lemma 6.1. *Let $F:\mathscr{P}_m\to\mathbb{R}$ be a functional defined and having Frechet derivative $F'(P)$ for all $P\in\mathscr{P}_m$. If $F(P^*)$ is a local extrema of $F(P)$ then $F'(P^*):\mathscr{P}_m\to\mathbb{R}$ is the zero functional.*

Proof. Take $\mathscr{S}=\mathscr{K}=\Sigma=\mathscr{P}_m$ in Theorem 1.7 in [6, p. 34]. Also, use the norm $\|\sum_{j=0}^m a_jz^j\|=\sum_{j=0}^m|a_j|$ on $\mathscr{P}_m$. Q.E.D.

By computing the Frechet derivative for $\rho_\Phi(f,P)$, we obtain necessary conditions for polynomial best approximation.

THEOREM 6.2. *Let D_Φ satisfy* (2.6) *and* (2.7) *and let $\Phi'(u)$ be continuous in* $[0, \infty)$ *and suppose*

$$K_1 = \sup\{u^{1/2}\Phi'(u): u \in [0, \infty)\} < \infty. \tag{6.1}$$

Further assume

$$\int\int |z|^l \, d\mu < \infty, \qquad l = 0, 1, 2 \cdots 2m.$$

Let $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ be measurable and satisfy (2.10). *Then*

$$\int\int \Phi'(|f - P^*|^2) \operatorname{Re}\{\overline{(f - P^*)}\, Y\} \, d\mu = 0$$

for all $Y \in \mathscr{P}_m$, $P^* \in \mathscr{B}_{m0\Phi}(f)$.

Proof. Define $F(P) = \rho_\Phi(f, P)$ for all $P \in \mathscr{P}_m$, and fixing $P \in \mathscr{P}_m$, define

$$G(Y) = -2 \int\int \Phi'(|f - P|^2) \operatorname{Re}\{\overline{(f - P)}\, Y\} \, d\mu \qquad \text{for all } Y \in \mathscr{P}_m.$$

We shall show $G(Y)$ is the Frechet derivative of $F(P)$—the result then follows from Lemma 6.1. In doing so, we use the same norm on $\mathscr{P}_m$ as in Lemma 6.1. Let $z \in \mathbb{C}$ and $\delta P \in \mathscr{P}_m$. By Taylor's theorem,

$$\Phi(|f(z) - (P(z) + \delta P(z))|^2) = \Phi(|f(z) - P(z)|^2) + \Phi'(|\zeta(z)|^2)\, \Delta z,$$

where $\zeta(z)$ lies on the line segment between $f(z) - P(z)$ and $f(z) - (P(z) + \delta P(z))$ (with the obvious interpretation when $|f(z)| = \infty$) and where Δz may be written as

$$\Delta z = -2 \operatorname{Re}\{\overline{(f(z) - P(z))}\, \delta P(z)\} + |\delta P(z)|^2.$$

Define $\psi(z) = \Phi'(|z|^2)\, \bar{z}$, $z \in \mathbb{C}$, and $\psi(\infty) = 0$. We see that

$$\begin{aligned}
&|F(P + \delta P) - F(P) - G(\delta P)| \\
&\quad = \Bigg| 2 \int\int \operatorname{Re}\{[\psi(f(z) - P(z)) - \psi(\zeta(z))]\, \delta P(z)\} \, d\mu(z) \\
&\qquad + \int\int \Phi'(|\zeta(z)|^2)[2 \operatorname{Re}\{\overline{[\zeta(z) - (f(z) - P(z))]}\, \delta P(z)\} \\
&\qquad + |\delta P(z)|^2] \, d\mu(z) \Bigg| \\
&\quad < 2 \|\delta P\| \int\int |\psi(f(z) - P(z)) - \psi(\zeta(z))| \max\{1, |z|^m\} \, d\mu(z) \\
&\qquad + 3 \|\delta P\|^2 \sup\{|\Phi'(u)|: u \in [0, \infty)\} \int\int \max\{1, |z|^{2m}\} \, d\mu(z).
\end{aligned} \tag{6.2}$$

Here we have used the inequality $|\delta P(z)| \leqslant \|\delta P\| \max\{1, |z|^m\}$. Now $|\psi(z)| \leqslant K_1$, $z \in \mathbb{C} \cup \{\infty\}$, by (6.1). Further for all $z \in \mathbb{C}$, $\psi(\zeta(z)) \to \psi(f(z) - P(z))$ as $\|\delta P\| \to 0$ (even if $|f(z)| = \infty$). By Lebesgue's Dominated Convergence Theorem, the first integral in the right member of (6.2) $\to 0$ as $\|\delta P\| \to 0$. This [6, p. 25] shows that $G = F'(P)$ and the result follows.

Q.E.D.

Remarks. (i) For $\Phi(u) = u^\alpha(1 + u^\alpha)^{-1}$, $u \in [0, \infty]$, we note that (6.1) holds if $\alpha \geqslant 1$ but not if $\alpha > 1$. The result can be slightly strengthened to allow $\alpha > 1/2$.

(ii) It is possible to obtain a second necessary condition, using $\Phi''(u)$, but we omit the details.

7. Convergence

Under mild analyticity restrictions on f, we can show $E_{mnD}(f) \to 0$ when $m \to \infty$ or $n \to \infty$ and that sequences of best approximations converge in measure (μ) to f.

THEOREM 7.1. *Let $\mathscr{C} = \operatorname{supp}[\mu] \backslash \mathscr{E}$, where $\mathscr{E}$ is a closed set of* meas 0. *Let f be continuous in $\mathscr{C}$ and analytic in its interior. Let $m(1), m(2) \cdots, n(1), n(2) \cdots$ be non-negative integers such that either $m(k) \to \infty$ or $n(k) \to \infty$. Then*

(i) $E_{m(k)n(k)D}(f) \to 0$.

(ii) *If $R_k \in \mathscr{R}_{m(k)n(k)D}(f)$ $(k = 1, 2, \ldots)$, then $R_k \to f$ in measure* (μ).

Proof. We can clearly assume $f(z)$ satisfies (2.10) since altering its values in $\mathbb{C} \backslash \operatorname{supp}[\mu]$ does not affect any $\mathscr{R}_{mnD}(f)$.

(i) Suppose, first, $m(k) \to \infty$. Let $\varepsilon > 0$. Choose $r > 0$, $\delta > 0$ such that

$$\mu\{z: |z| \geqslant r\} < \varepsilon/3 \tag{7.1}$$

and

$$\mu\{z: d(z, \mathscr{E}) < \delta\} < \varepsilon/3, \tag{7.2}$$

where $d(z, \mathscr{E}) = \min\{|z - u|: u \in \mathscr{E}\}$. Then, $\mathscr{C}^* = \{z: |z| > r\} \cup \{z: d(z, \mathscr{E}) < \delta\} \cup (\mathbb{C} \backslash \operatorname{supp}[\mu])$ is open and so has at most denumerably many components—$\mathscr{C}_1, \mathscr{C}_2, \ldots, \mathscr{C}_\infty$, where $\mathscr{C}_\infty$ is the unbounded one. Join $\mathscr{C}_1$, $\mathscr{C}_2 \cdots$, respectively, to $\mathscr{C}_\infty$ by open rectangles $\mathscr{R}_1, \mathscr{R}_2 \cdots$ such that

$$\mu(\mathscr{R}_j) < \varepsilon 2^{-j}/3, \qquad j = 1, 2 \cdots. \tag{7.3}$$

Then $\mathscr{D} = \mathscr{C}^* \cup (\bigcup_{j=1}^{\infty} \mathscr{R}_j)$ is a connected open set;

$$\mathscr{K} = \mathbb{C} \backslash \mathscr{D} = \left\{ z : |z| \leqslant r, d(z, \mathscr{E}) \geqslant \delta, z \in \operatorname{supp}[\mu], \right.$$

$$\left. \text{and } z \notin \bigcup_{j=1}^{\infty} \mathscr{R}_j \right\}$$

is compact, and f is continuous on $\mathscr{K}$ and analytic in its interior (as $\mathscr{K} \subset \mathscr{C}$). By Mergelyan's theorem [7, p. 367], there are polynomials P_1, $P_2 \cdots$ such that $P_j \to f$ uniformly in $\mathscr{K}$ and so $D(P_j(z), f(z)) \to 0$ in $\mathscr{K}$. Using (7.1), (7.2), (7.3), we see that $\mu(\mathbb{C} \backslash \mathscr{K}) < \varepsilon$ and, hence $\limsup_{j \to \infty} \rho_D(P_j, f) < \varepsilon$. As $\varepsilon > 0$ is arbitrary, $E_{m(k)n(k)D}(f) \to 0$.

Next, suppose $n(k) \to \infty$. Let $\varepsilon > 0$. If $\mathscr{F}(\delta) = \{z \in \mathscr{C} : f(z) = -\delta\}$ then $\{\mathscr{F}(\delta) : \delta \in [0, \varepsilon]\}$ is an uncountable family of disjoint Borel sets. Given a positive integer l, at most finitely many $\mathscr{F}(\delta)$ can satisfy $\operatorname{meas}(\mathscr{F}(\delta) \cap \{z : |z| \leqslant l\}) \geqslant 1/l$. We deduce $\operatorname{meas}(\mathscr{F}(\delta)) = 0$ for all but at most denumerably many $\delta \in [0, \varepsilon]$. Choose now $\delta \in [0, \varepsilon)$ for which $\operatorname{meas}(\mathscr{F}(\delta)) = 0$ and set $g(z) = (f(z) + \delta)^{-1}$. Then $\tilde{\mathscr{E}} = \mathscr{E} \cup \mathscr{F}(\delta)$ is closed and has meas 0. Further we see easily that $g(z)$ is continuous in $\mathscr{C} \backslash \tilde{\mathscr{E}}$ and analytic in its interior. We can, as in the first part of the proof, construct a compact subset $\tilde{\mathscr{K}}$ of $\mathscr{C} \backslash \tilde{\mathscr{E}}$ such that $g(z)$ and $f(z)$ are continuous in $\tilde{\mathscr{K}}$ and analytic in its interior, $\mu(\mathbb{C} \backslash \tilde{\mathscr{K}}) < \varepsilon$ and $\mathbb{C} \backslash \tilde{\mathscr{K}}$ is connected. Mergelyan's theorem yields polynomials $\tilde{P}_1$, $\tilde{P}_2 \cdots$ satisfying $\tilde{P}_j(z) \to g(z)$, uniformly in $\tilde{\mathscr{K}}$. As $f(z)$ is continuous in $\tilde{\mathscr{K}}$, $g(z)$ has no zeroes there. Further $\delta < \varepsilon$. Thus for some j,

$$|f(z) - 1/\tilde{P}_j(z)| < \varepsilon \qquad \text{for all } z \in \tilde{\mathscr{K}}; \quad \mu(\mathbb{C} \backslash \tilde{\mathscr{K}}) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we have shown that there are polynomials $Q_1, Q_2 \cdots$ such that $Q_j^{-1} \to f$ in meas(μ). By Lemma 3.1, $\rho_D(f, Q_j^{-1}) \to 0$. It follows that $E_{m(k)n(k)D}(f) \to 0$.

(ii) Follows from Lemma 3.1 and (i). Q.E.D.

Remarks. (i) It is interesting to compare Theorem 7.1 to Padé convergence theorems. When f has singularities of positive (logarithmic) capacity, its Padé approximants need not converge in measure in any neighborhood of zero (no matter how large is the power series' finite radius of convergence); see [5]. Similar counterexamples hold for more general rational interpolatory (and best) approximations. Theorem 7.1 shows that our best approximations converge in measure (μ) and hence (locally) in meas, if meas is absolutely continuous with respect to μ—subject only to f having singularities of at most meas 0 in $\mathbb{C}$. Thus, f can (in an obvious sense) have denumerably many natural boundaries and Theorem 7.1 would still be applicable.

(ii) Theorem 7.1 raises the possibility of a converse theorem: If $E_{mnD}(f) \to 0$ whenever $\max\{m, n\} \to \infty$ and if $f: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is measurable and satisfies (2.10), then does f have singularities of at most meas 0?

(iii) Other natural questions arise: If $\operatorname{supp}[\mu] \neq \mathbb{C}$, do the poles of best rational approximations lie in $\mathbb{C} \backslash \operatorname{supp}[\mu]$? Is every singularity of f a limit point of poles of best approximations? Is there an analogue of the de Montessus de Balloré Theorem [2]?

8. Estimating $E_{mnD}(f)$

Explicit information on μ, D and f allows estimation of $E_{mnD}(f)$ in certain cases.

THEOREM 8.1. *Let* $\Phi(u) = (u^\alpha(1+u^\alpha)^{-1})^\beta$, $u \in [0, \infty]$, *where* $\alpha, \beta > 0$. *Let* D_Φ *satisfy* (2.6) *and* (2.7). *Let* $\mu(\mathscr{U}) = c\Pi^{-1} \iint_{\mathscr{U}} \exp(-c(x^2+y^2))\,dx\,dy$ *for all Borel sets* $\mathscr{U}$ *where c is a positive constant. Thus*

$$\rho_\Phi(g, h) = c\Pi^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{|g(z)-h(z)|^{2\alpha}(1+|g(z)-h(z)|^{2\alpha})^{-1}\}^\beta \times \exp(-c(x^2+y^2))\,dx\,dy$$

($z = x+iy$) *for all measurable* $g, h: \mathbb{C} \to \mathbb{C} \cup \{\infty\}$.

Let $f: \mathbb{C} \to \mathbb{C}$ *be entire of order* $\rho < 2$. *Then*

$$\limsup_{m \to \infty} \{E_{m0\Phi}(f)\}^{1/(m \log m)} \leqslant \exp(\alpha\beta(1-2\rho^{-1})) < 1. \tag{8.1}$$

Taking, instead, $D = D_1$ *as in* (2.4), (2.5) *we have*

$$\limsup_{m \to \infty} \{E_{m01}(f)\}^{1/m \log m} \leqslant \exp(1-2\rho^{-1}) < 1.$$

Proof. Choose $1/2 < \delta < \rho^{-1}$ and $\varepsilon > 0$ such that $\Delta = [\rho(1+\varepsilon)]^{-1} - \delta > 0$. Writing $f(z) = \sum_{j=0}^\infty a_j z^j$, we have [4, Theorem 14.2, p. 186],

$$\limsup_{j \to \infty} |a_j|^{1/(j \log j)} = \exp(-\rho^{-1}). \tag{8.2}$$

Then, for large n, $|a_n|\, n^{\delta n} \leqslant \exp(-n(\log n)\Delta) = n^{-n\Delta}$. Setting $P_m(z) = \sum_{j=0}^m a_j z^j$, we have, for all large m,

$$\max\{|f(z)-P_m(z)|: |z| \leqslant m^\delta\} \leqslant \sum_{j=m+1}^{\infty} j^{-j\Delta} \leqslant m^{-m\Delta}. \tag{8.3}$$

Further,

$$\mu\{z: |z| \geqslant m^{\delta}\} = \exp(-cm^{2\delta}). \tag{8.4}$$

So

$$\begin{aligned} \rho_{\Phi}(f, P_m) &\leqslant \int\int_{\{z: |z| \leqslant m^{\delta}\}} |f(z) - P_m(z)|^{2\alpha\beta} \, d\mu + \mu\{z: |z| \geqslant m^{\delta}\} \\ &\leqslant m^{-m\Delta 2\alpha\beta} + \exp(-cm^{2\delta}). \end{aligned}$$

Here we have used $\Phi(u^2) \leqslant u^{2\alpha\beta}$ all $u \geqslant 0$ and (8.3), (8.4). As $\delta > 1/2$, we deduce

$$\limsup_{m \to \infty} \{E_{m0\Phi}(f)\}^{1/(m \log m)} \leqslant \exp(-2\Delta\alpha\beta).$$

Finally, as Δ can be made arbitrarily close to $\rho^{-1} - 2^{-1}$, (8.1) follows. Similarly for D_1. Q.E.D.

Observe the similarity between (8.1) and (8.2).

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